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COMMENTS ON CONFINEMENT CRITERIA †

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For a QED_2 model with $SU(n)$ flavour, the nature of the physical-states space is more subtle than one expects on the basis of the loop criterion for confinement. One may have colour confinement without confinement of the fundamental flavour representation. We also discuss attempts to formulate confinement criteria in which the quark fields play a more fundamental role.

1. Introduction

The standard picture for the possible mechanism for quark confinement which originates from ideas of Wilson [1] and has been elaborated by several authors [2], uses mainly quasi-classical ideas and leads to certain consequences which partially justify a potential theoretical picture of spatially increasing interaction between "external" quarks. The support comes from lattice investigations [1,3,4] on the one hand and certain qualitative assumptions on the Callan-Symanzik $\beta(g)$ functions [4] (assuming that such functions globally exist). Among the widely-accepted consequences of this "test-charge" picture, the Wilson loop criterion plays a significant role. One considers the Wilson (Euclidean) loop integral over the vector potential

$$\langle \exp(ie \oint A_\mu \, \mathrm{d}x^\mu) \rangle \tag{1}$$

and studies the dependence of its logarithm on the extension of the enclosed surface. If this quantity is proportional to the area of the enclosed surface, one argues that this leads to quark confinement (*via* an energy increase with the separation distance between "quark probes"), whereas a linear dependence on the extension (i.e.,

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the diameter of a square surface) yields the unconfining situation.

Within the continuous field theory framework the support for the loop criterion comes from the Schwinger model, i.e., QED_2 either in its massless- or massive-fermion version. A free A_μ field coupled to external charges leads to a surface increase of (1), whereas the direct computation of the model with quantum ψ's shows the absence of charged states in the physical-state space of the solution. Two aspects of this model are remarkable: first the loop-confining criterion only works with test charges in a theory which contains *only* gluon fields. The test fails if quark fields are present, the logarithm of the loop integral in that case is easily demonstrated to be proportional to the linear dimensions. This fact seems to be implicitly known in the existing literature. It is sometimes argued that test charges are equivalent to infinitely heavy quarks. However, this philosophy taken literally (say in the massive QED_2) meets some obstacles and its validity in quantum field theory has never been rigorously established. The second aspect of the loop picture in two-dimensional gauge theories, namely the implication of quark confinement (i.e., the non-appearance of the fundamental representation of flavour in the physical spectrum), is not borne out by field theoretical reality of massless QED_2 with SU(2) (isospin) flavour. As we will show in sect. 2, this model contains colourless physical states with half-integer isospin. The introduction of a quark mass partially suppresses this phenomenon, however, for a particular value of one of the parameters of the theory (the θ angle $\theta = \pi$) colourless $I = \frac{1}{2}$ states will appear. These true bound states can be viewed as the relics of the $I = \frac{1}{2}$ physical states in the massless theory. In this way we are able to shed some light on one of Coleman's open problems [5]. It is conceivable that in QCD_4 theories the quark fields are playing an even more essential role than in QED_2. For example it may turn out that the loop test in the continuous field theory does not give the desired surface proportionality and yet the colour of quantum quarks may be confined. If this turns out to be a relevant point in future discussions, one would have to consider other criteria for confinement. Sect. 3 discusses this point in an (unfortunately) very sketchy and speculative way.

We would like to stress that by colour "confinement" we merely mean the absence of coloured states from the physical (gauge-invariant) spectrum. Our use of this terminology does not include other physically important properties as statements on Regge trajectories and the relation to dual-string models [16] and properties of hadronic structure which to some extend are implicitly accounted for in the Wilson criterion [1].

2. Model for unconfined fundamental flavour

The confinement picture based on increasing potentials or on bilocal (or multilocal) gauge-invariant strings leads to the disappearance of the fundamental flavour representation from the physical states. As a simple illustration imagine an SU(n),

$D = 2$ Schwinger model with Abelian gauge coupling. The non-confined physical states are expected to be "mesons" (i.e., they have ψ charge zero) of isospin $I = 0, 1, \ldots$. This conclusion is, however, false; the physical sector rather contains also $I = \frac{1}{2}$ states. In other words, field theory leads to a more economical confinement mechanism than potential theory, only the charges coupled to gauge fields are confined, the fundamental flavour ($I = \frac{1}{2}$), appears among the physical states.

In the following we give a simple demonstration of this statement. In analogy to the ordinary Schwinger model we make the SU(n) ansatz (unitary gauge of ref. [6]):

$$\psi^l(x) = e^{i\chi^{(+)}(x)}\psi_0^l(x)\, e^{i\chi^{(-)}(x)} , \tag{2}$$

$$\chi(x) = \alpha j(x) + \beta\gamma^5 \tilde{j}(x) + ia\, \gamma^5 \Sigma(x) , \tag{3}$$

$$j(x) = j_L(u) + j_R(v) , \qquad \tilde{j}(x) = j_L(u) - j_R(v) ,$$

$$\langle j_L(u) j_L(u')\rangle = -\frac{n}{4\pi} \{\ln(u - u' - i\epsilon) + i\tfrac{1}{2}\pi\} ,$$

$$\langle j_R(v) j_R(v')\rangle = -\frac{n}{4\pi} \{\ln(v - v' - i\epsilon) + i\tfrac{1}{2}\pi\} ,$$

$$u = t + x , \qquad v = t - x , \tag{4}$$

and Σ is a free field of mass m.

From the validity of the Dirac and Maxwell equations:

$$i\gamma^\mu \partial_\mu \psi^l(x) + \lim_{\substack{\epsilon\to 0 \\ \epsilon^2 \neq 0}} \tfrac{1}{2} e\gamma^\mu \{A_\mu(x + \epsilon)\, \psi^l(x) + \psi^l(x)\, A_\mu(x - \epsilon)\} = 0 , \tag{5}$$

$$\partial_\nu F^{\mu\nu} = -eJ^\mu(x) , \tag{6}$$

we obtain

$$eA_\mu(x) = (\alpha - \beta)\, \partial_\mu j(x) - a\, \epsilon_{\mu\nu}\partial^\nu \Sigma(x) , \tag{7}$$

$$\beta = \frac{\sqrt{\pi}}{n} , \qquad m = \sqrt{\frac{n}{\pi}}\, e . \tag{8}$$

In working out the Maxwell equation we have used the gauge-invariant definition

$$J^\mu(x) = - \lim_{\substack{\epsilon\to 0 \\ \epsilon^2 \neq 0}} \mathrm{Tr}[\gamma^0\gamma^\mu(T(x + \epsilon, x) - \langle T(x + \epsilon, x)\rangle)] , \tag{9}$$

where

$$T(x, y) = e^{iK^{(+)}(x,y)}\, \psi_0(x)\, \psi_0^\dagger(y)\, e^{iK^{-}(x,y)} , \tag{10}$$

$$K^{(\pm)}(x,y) = e \int_x^y A^{\mu^{(\pm)}} \mathrm{d}z_\mu + \chi^{(\pm)}(x) - \chi^{(\pm)}(y) . \tag{11}$$

This leads to

$$J^\mu(x) = \left(1 - \frac{n}{\sqrt{\pi}} \beta\right) \partial^\mu j - \frac{na}{\pi} \epsilon^{\mu\nu} \partial_\nu \Sigma(x) , \tag{12}$$

and, therefore, *via* (7) to (6).

The value $a = \sqrt{\pi}/n$ leads to the canonical short-distance behaviour of gauge-invariant composites expected from the super-renormalizability of the formal Lagrangian

In contrast to the U(1) Schwinger model there is no value of the gauge parameter α such that

$$\psi^i(x) = e^{i(\sqrt{\pi}/n)\gamma^5 \Sigma(x)} \sigma^i , \tag{13}$$

where σ is a true spurion, i e., a field with zero dimension and spin The "optimal" gauge fulfils

$$\dim \sigma^i = \mathrm{spin}\, \sigma^i = \frac{1}{2}\left(1 - \frac{1}{n}\right) \tag{14}$$

The word "optimal" has the precise meaning that only for this particular gauge

$$[\sigma^i, J_\mu(x)] = 0 . \tag{15}$$

The operator σ^k still depends on the SU(n) vector current, i.e ,

$$[\sigma^k, J^i_\mu] \neq 0 , \tag{16}$$

with

$$J^i_\mu = \overline{\psi}_0 \gamma_\mu \tfrac{1}{2}\lambda^i \psi_0 = j^i_\mu \tag{17}$$

This free current is the result of the gauge-invariant bilocal limiting procedure. The operator algebra of the physical Hilbert space contains exponential functions of Σ (no infra-red problem since $m \neq 0$) and, hence, by a limiting procedure also the σ's. They generate physical states (without mass gap) which carry the fundamental flavour but are colourless

Now we consider the case of massive QED_2, i.e., the massive version of the Schwinger model. In that case one may introduce into the free spinor Lagrangian a "dummy" Lagrangian parameter θ by making a chiral transformation

$$M\overline{\psi}\psi \to M\,\overline{\psi}\psi + iM\,\overline{\psi}\gamma^5\psi \sin\theta \tag{18}$$

The content of the free theory (i e , the existence of two discrete symmetries P and C) is not changed This is even more visible in the "bosonized" version

$m\cos(2\sqrt{\pi}\phi) \to m\cos(2\sqrt{\pi}\phi+\theta)$ However, the switching-on of the electromagnetic field converts it into a relevant Lagrangian parameter (a somewhat different way is to introduce θ as an apparent non-Lagrangian parameter through cosmological boundary behaviour of the electric field, see Coleman et al. [7]). In an SU(2) theory this procedure leads in the bosonized description to [5] ($m = \sqrt{2/\pi}\, e$)

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\gamma^\mu\phi - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\,\partial_\mu\phi_3\partial^\mu\phi_3$$

$$+ M\cos(\sqrt{2\pi}\,\phi+\theta)\cos(\sqrt{2\pi}\,\phi_3)\,, \tag{19}$$

where ϕ_3 is the isovector pseudoscalar potential. The natural U(1) neutral SU(2) non-trivial states are described by (we take a bosonic realization of these solitons) applying

$$e^{i\sqrt{2\pi}\int_{-\infty}^{x}\phi_3\,dx'} \tag{20}$$

to the vacuum

With

$$j_{\mu 3} = -\frac{1}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu\phi_3\,, \tag{21}$$

these states are members of an isotriplet. However, for $\theta = \pi$ we have the following interesting possibility: we use the states generated by the soliton operator with the smaller coefficient leading to the third component of isospin $\frac{1}{2}$.

$$e^{i\sqrt{\pi/2}\int_{-\infty}^{x}\phi_3\,dx'}\,. \tag{22}$$

This in itself would not lead to a finite energy state, however multiplying it with a kink operator $K_\phi(x)$ in ϕ, similar to the quantum kink of the ϕ_2^4 theory, we do obtain a finite-energy state in the form

$$K_\phi(x)\,e^{i\sqrt{\pi/2}\int_{-\infty}^{x}\phi_3\,dx'}|0\rangle\,. \tag{23}$$

An explicit description of K can given in terms of a properly modified Bogoliubov transformation [8]. The only relevant property for our present use is that K_ϕ "rotates" ϕ into $-\phi$ and therefore compensates the undesired change of sign arising from using (22) instead of (23).

So the mass term acts like a filter for neutral $I_3 = \frac{1}{2}$ states, only for $\theta = \pi$ do those states pass and come out The appearance of $I = \frac{1}{2}$ physical states in the massless Schwinger model is the explanation for one of Coleman's problems at the end of his paper [5].

3. Confinement and bilocals, spurionization of colour

The short coming of the loop criterion discussed in the previous section and the possibly more dominant role of quantized quark fields in $D = 4$ makes it desirable to find alternative tests for colour confinement.

In two-dimensional theories of colour confinement the gauge-invariant bilocals ("dipole operators"),

$$\psi(x) \exp(ie \int_x^y A_\mu \, dz^\mu) \, \psi^\dagger(y) \,, \tag{24}$$

turn out to be very useful [9]. For the non-Abelian case the A_μ are matrices, the integral is path-ordered and a trace over colour indices is to be taken. The short-distance problem involved in the ordering of the exponential and the handling of the end-point singularities is simple, thanks to the super-renormalizability of $D = 2$ gauge theories.

Let us now look at the behaviour of the gauge-invariant observable $\mathcal{O}$ in gauge-invariant "dipole states":

$$\langle 0|D^\dagger(x, y)\, \mathcal{O}(z)\, D(x, y)|0\rangle \,. \tag{25}$$

In order to talk about normalizable states we should perform the well-known quantum field theoretical smearing in x and y which we will not do in order to keep our formulas simple.

The existence of charged sectors would then come about from the asymptotic behaviour of neutral states: apart from small-distance fluctuations the + and − charges are expected to stabilize at the ends of the dipole if the separation increases and the expectation values are expected to converge in the following sense:

$$\langle D^\dagger(x, y)\, \mathcal{O}(z)\, D(x, s)\rangle \underset{y\to\infty}{\sim} \langle\psi(x)|\mathcal{O}(z)|\psi(x)\rangle\, F(\text{path}) \,. \tag{26}$$

The second factor contains all the complicated dependence on how one goes with y "behind the moon"; all the $\mathcal{O}$ dependence is contained in the first factor which describes the $\mathcal{O}$ expectation value in some state tentatively written as $\psi|0\rangle$.

This picture of the reconstruction of charge sectors in a theory of neutral dipoles in an extrapolation of the results on sector construction in the algebraic field theory for gauge invariance of the first kind [10].

The most prominent observables $\mathcal{O}$ which one may be interested in are the charge density, the energy density and their fluctuations:

$$\mathcal{O}(z) = \begin{cases} j_0(z), j_0(z_1)\, j_0(z_2) \,, \; \ldots \\ \mathcal{H}(z) \,, \mathcal{H}(z_1)\, \mathcal{H}(z_2) \,, \; \ldots \end{cases} \tag{27}$$

On the other hand, in the presence of colour confinement, one expects *no stabilization* for increasing distances. In particular the expectation value of $j_0(z)$ should reveal that charges do not stabilize at the end-points of strings. This picture of non-confinement *versus* confinement works for all known two-dimensional models. Consider the bilocals in the massive (or massless) Schwinger model [9]

$$D(x, y) = N(x-y) \exp\{i\sqrt{\pi}(\gamma^5 \phi(x) - \int_x^y \epsilon^{\mu\nu} \partial_\nu \phi \, dx'_\mu - \gamma^5 \phi(y))\} ,$$

$$N(z) = -\frac{1}{2\pi} \begin{bmatrix} \dfrac{-i}{z^0 + z^1} & c \\ c & \dfrac{-i}{z^0 - z^1} \end{bmatrix} . \tag{28}$$

With this N the exponential is normal ordered as a whole (of course different ordering prescriptions correspond to different N's). Here ϕ is a solution of the *massive* (mass m) sine-Gordon equation.

The electromagnetic field and the current expressed in terms of ϕ are

$$j_\mu = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \, \partial^\nu \phi ,$$

$$F_{\mu\nu} = m \, \epsilon_{\mu\nu} \, \phi . \tag{29}$$

It is known that the massive sine-Gordon model does not lead to charge sectors. Technically speaking, this means that the dipole states do not approach a stable limit. Only in the case $m \to 0$, which is equivalent to the switching-off of electromagnetism, the dipole states converge and the $\psi|0\rangle$ in formula (26) is a local charged field which generated the topological charge sectors of the Sine-Gordon equation.

Concerning the generalization of these ideas to $D = 4$ gauge theories one has to pay attention to avoid any artificially generated confinement. A bilocal state of the form (36) would be in agreement with Gauss' law (the electric flux goes through a line of thin tube). However, forcing all the flux through a thin tube is expected to lead to infinite-energy states in the limit $(x - y) \to \infty$ which have nothing to do with a natural confinement mechanism. Here one should allow for more general flux pictures in the dipole operators (trace over colour indices):

$$D(x, y) = \psi(x) \exp(ie \int G^\mu(x, y, z) \underset{\sim}{A}_\mu(z) \, d^4z) \, \psi^\dagger(y) . \tag{30}$$

For $t_x = t_y$ one may think of a "Coulombic" shape of the kernel G:

$$G^\mu = \begin{cases} 0 & \mu = 0 \\ G^l(\boldsymbol{x},\boldsymbol{y};\boldsymbol{z}) & \delta(t_z - t_x) , \end{cases}$$

$$G(\boldsymbol{x}, \boldsymbol{y}; z) = -\nabla_z \left(\frac{1}{|x - z|} - \frac{1}{|y - z|} \right) . \tag{31}$$

With such a general form of the charge "backflow" between x and y we obtain gauge-invariant dipoles in terms of which the confinement discussion can be more naturally formulated. Note that this consideration is very formal, we have not spelled out which ordering will lead to a gauge-invariant result [13] The formulation of a gauge-invariant "bilocal" in the non-Abelian theory with A_μ fluxes different from line integrals is presently an open problem.

Again we expect the colour-charge instability at the end-points and the increase of energy and energy fluctuations to be the signals for confinement

Therefore this intuitively reasonable picture of confinement is a good candidate for a confinement criterion in two-dimensional gauge theories. In general it may not be a very manageable criterion A formula in terms of Euclidean functional integrals using concepts of integration over winding gauge classes or instanton solutions [11] may be preferable Here a simplifying idea may be the understanding of confinement in terms of colour "spurionization" [12] (or condensation in analogy with the BCS theory) expressed by the non-vanishing gauge-invariant 1-point function [U(1) flavour]

$$\langle\hat{\psi}\rangle \neq 0\,, \qquad \hat{\psi}(x) = \psi(x)\, \mathrm{e}^{ie\int_x^\infty A_\mu^{\mathrm{L}}\, \mathrm{d}z^\mu}\,, \tag{32}$$

where A_μ^{L} is the longitudinal part of the gauge field. The decomposition of A_μ into A_μ^{T} and A_μ^{L} makes perfect sense in the Euclidean functional integral. The A_μ configuration giving the non-vanishing contribution has non-integer winding number [12] For non-trivial flavour groups, say SU(N), the spurionization picture is more complicated (see discussion of sect 2) and one obtains additional insight by comparing this model with the simple U(1) model.

The concept of "colour spurionization" which we are using here requires some remarks Assume that we already know the vanishing of the "gaussian" colour charge $Q^{\mathrm{c}} = \int j_0^{\mathrm{c}}\, \mathrm{d}_x^d$ on physical states Our formalism also contains a "counting" colour charge coming from the gauge invariance of the first kind.

$$\psi \to U\,\psi\,, \qquad U \in \text{colour group} \tag{33}$$

This counting charge should be physically irrelevant, i.e. the description should have redundant elements. Indeed in the Schwinger model the gauge-invariant (in the sense of local 2nd-kind transformations) $\hat{\psi}$ has the representation

$$\hat{\psi}(x) \equiv \psi(x)\, \mathrm{e}^{ie\int_x^\infty A_\mu^{\mathrm{L}} \mathrm{d}z^\mu} = \mathrm{e}^{i\sqrt{\pi}\gamma^5\Sigma(x)}\, \sigma\,, \tag{34}$$

where the x-independent unitary spurion operator σ carries the counting charge and necessarily leads to an infinite vacuum degeneracy. The unique vacuum description corresponding to a diagonalization of σ makes the breaking of the gauge invariance of the first kind (i.e. the counting charge) manifest *via* non-vanishing expectation values (32). Looking for signals of colour confinement *via* colour spurionization, therefore, means looking for non-vanishing expectation values of quantities which formally change under constant gauge transformations but are locally (due to A_μ fluxes to infinity) gauge-invariant.

Recently arguments for the possibility of transmutation of "internal" (colour) spin into space spin have been given [14]. A spurionization as (32) which requires in some sense the opposite mechanism does not appear to us as an absurd possibility in four-dimensional quantum field theories It is also conceivable that only colour but not spin "spurionizes". For example, (for trivial flavour) the space spin may be carried by self-dual Majorana fields in which case one would like to show

$$\langle \hat{\psi} \hat{\psi} \rangle \neq 0 \,,$$

where $\hat{\psi}$ is some gauge-invariant $\hat{\psi}$ similar to the one used in (33). An understanding of such colour spurionization phenomena in physical (Minkowski) space presumably requires better insight into the field theory of monopoles. Such speculation is based on the fact that the understanding of two-dimensional "bosonization" of charged fields has turned out to be helpful for the QED_2 confinement problem. On the other hand, two-dimensional confining models can be very efficiently understood in terms of "unusual" field configurations [12]. It is in this direction that we entertain some hopes for the understanding of $D = 4$ colour confinement

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